

STRUCTURE THEOREMS FOR BASIC ALGEBRAS

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ABSTRACT. A basic finite dimensional algebra over an algebraically closed field k is isomorphic to a quotient of a tensor algebra by an admissible ideal. The category of left modules over the algebra is isomorphic to the category of representations of a finite quiver with relations. In this article we will remove the assumption that k is algebraically closed to look at both perfect and non-perfect fields. We will introduce the notion of species with relations to describe the category of left modules over such algebras. If the field is not perfect, then the algebra is isomorphic to a quotient of a tensor algebra by an ideal that is no longer admissible in general. This gives hereditary algebras isomorphic to a quotient of a tensor algebra by a non-zero ideal. We will show that these non-zero ideals correspond to cyclic subgraphs of the graph associated to the species of the algebra. This will lead to the ideal being zero in the case when the underlying graph of the algebra is a tree.

CONTENTS

| | |
|---|----|
| 1. Preliminaries | 2 |
| 2. Species | 4 |
| 3. Basic Algebras over Perfect Fields | 7 |
| 4. Basic Finite Dimensional Split Algebras | 12 |
| 5. Hereditary basic finite dimensional split algebras | 15 |
| References | 19 |

It is well known that a basic finite dimensional algebra Λ over an algebraically closed field k is isomorphic to a quotient of a path algebra $k\Gamma$ of a finite quiver Γ . Moreover the path algebra $k\Gamma$ is isomorphic to a tensor algebra [ARS, Theorem III.1.9]. This was first outlined by Gabriel in [Ga1], and he gave a concise proof in [Ga2, Section 4.3]. From now on we will call this result *Gabriel's structure theorem for basic finite dimensional algebras over an algebraically closed field*, or just the *structure theorem*.

In this article we will discuss what happens if the field k is not algebraically closed. If one tries to follow the proof of Gabriel, two assumptions on the algebra Λ arise; the first is that Λ splits, i.e. the natural projection onto the quotient algebra $\pi: \Lambda \rightarrow \Lambda/\text{rad } \Lambda$ splits as a k -algebra homomorphism. Hence there exists an $\epsilon: \Lambda/\text{rad } \Lambda \rightarrow \Lambda$ such that $\pi \circ \epsilon \simeq \text{id}_{\Lambda/\text{rad } \Lambda}$. Via ϵ all Λ -modules can be viewed as $\Lambda/\text{rad } \Lambda$ -modules. The second assumption is that for any $\epsilon: \Lambda/\text{rad } \Lambda \rightarrow \Lambda$ such that $\pi \circ \epsilon \simeq \text{id}_{\Lambda/\text{rad } \Lambda}$ the short exact sequence

$$0 \rightarrow (\text{rad } \Lambda)^2 \rightarrow \text{rad } \Lambda \rightarrow \text{rad } \Lambda/(\text{rad } \Lambda)^2 \rightarrow 0$$

splits when it via ϵ is viewed as a sequence of $\Lambda/\text{rad } \Lambda - \Lambda/\text{rad } \Lambda$ -bimodules. If both these assumptions are fulfilled, we get a generalization of the structure theorem; Λ is isomorphic to a quotient of a tensor algebra, and this tensor algebra is constructed from a species associated to the algebra Λ [Ben, Proposition 4.1.10].

The main topic of this article is investigating what happens if we remove the second assumption above. We will show that Λ is still a quotient of a tensor algebra associated to a species, however not the same species as was used before: To get a morphism from a tensor algebra onto the algebra Λ we have to take the tensor algebra over a larger bimodule than the one used when both assumptions were fulfilled. Therefore the kernel of this morphism is no longer an admissible ideal in the tensor algebra, which gives some interesting observations in the case Λ is hereditary, e.g. the hereditary algebra Λ need no longer be a tensor algebra. Examples of such algebras are already known, and we will use an example from [DR2] to highlight this property.

In the first section we will introduce notions used throughout this article. Readers experienced with finite dimensional algebras will likely be familiar with all the notions introduced.

The second section introduces species with relations. Since the existing literature does not treat this concept in detail, we will give a fairly thorough discussion of it here.

In the third section we will give the structure theorem for finite dimensional basic split algebras for which the sequence

$$0 \rightarrow (\text{rad } \Lambda)^2 \rightarrow \text{rad } \Lambda \rightarrow \text{rad } \Lambda / (\text{rad } \Lambda)^2 \rightarrow 0$$

splits, using species with relations. Most results in this section are similar to well known results, but the usage of species with relations is however not common. We will also show that finite dimensional basic algebras over perfect fields satisfy the assumptions above.

The fourth section gives a structure theorem for finite dimensional basic split algebras. This structure theorem is a generalization of Gabriel's structure theorem, however it is not a generalization of the structure theorem given in section three.

In the last section we will describe hereditary basic finite dimensional split algebras. In contrast to the case for algebras over algebraically closed fields, the species of these hereditary algebras might have non-zero relations corresponding to subquivers for which the underlying graph contains cycles.

1. PRELIMINARIES

This first section will be used to introduce notions we will need in the rest of this article.

Throughout this section we will assume that Λ is an indecomposable finite dimensional algebra over a field k . Since Λ then is artinian, we know Λ/\mathfrak{r} is semisimple [La1, Theorem 4.14], where $\mathfrak{r} = \text{rad } \Lambda$ is the Jacobson radical of the algebra Λ . Since the algebra is finite dimensional, the radical \mathfrak{r} is nilpotent, i.e. $\mathfrak{r}^n = (0)$ for a large enough $n \in \mathbb{N}$.

When we view Λ as a left module over itself, it can be written as a direct sum of indecomposable projective left Λ -modules ${}_{\Lambda}\Lambda = \bigoplus_{i \in I} P_i$. When the P_i are pairwise non-isomorphic projective Λ -modules we say that Λ is *basic*. A finite dimensional algebra Λ is always Morita equivalent to a basic finite dimensional algebra. Thus, if we are interested in the module category of an algebra, we can always reduce the question to a basic algebra. If we assume that Λ is basic, then $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} P_i / \text{rad } P_i \simeq \bigoplus_{i \in I} D_i$, where D_i are division rings [DK, Theorem 3.5.4]. Since Λ was assumed to be a k -algebra, k will act centrally on the division rings D_i , i.e. for all $\lambda \in k$ and all $\lambda' \in D_i$ we have that $\lambda\lambda' = \lambda'\lambda$. The direct sum $\bigoplus_{i \in I} D_i$ contains k as a subfield.

We say that a finite dimensional k -algebra Λ is *elementary* if $\Lambda/\mathfrak{r} \simeq \bigoplus_{i=1}^n k$, i.e. isomorphic as a k -algebra to a finite direct sum of copies of k . If k is algebraically closed, i.e. k has no proper algebraic extension, then the only finite dimensional division algebra over k is k itself, so basic implies elementary when k is algebraically closed and Λ is finite dimensional.

Let R be a ring. An element $e \in R$ is called an *idempotent* if $e^2 = e$. We call two idempotents e and f *orthogonal* if $ef = 0 = fe$, and we call an idempotent e *primitive* if $e \neq f + g$ where f and g are nonzero orthogonal idempotents. A set of pairwise orthogonal primitive idempotents $\{e_1, e_2, \dots, e_n\}$ in a ring R will be called *complete* if $e_1 + e_2 + \dots + e_n = 1_R$, where 1_R is the multiplicative identity of R . Let Λ be a finite dimensional algebra and let 1_Λ be the multiplicative identity in Λ . Since ${}_\Lambda\Lambda = \bigoplus_{i \in I} P_i$, we will have $1_\Lambda = \sum_{i \in I} e_i$, where $e_i \in P_i$. It is easy to see that the elements $\{e_i\}_{i \in I}$ are pairwise orthogonal idempotents, and it can be shown that they are primitive [AF, Corollary 7.4]. Hence we have a complete set of pairwise orthogonal idempotents $\{e_i\}_{i \in I}$ in Λ , and we have ${}_\Lambda\Lambda = \bigoplus_{i \in I} \Lambda e_i$ where $\Lambda e_i \simeq P_i$ [AF, Corollary 7.3].

The quotient $\mathfrak{r}/\mathfrak{r}^2$ has a natural Λ/\mathfrak{r} -bimodule structure by letting $(\lambda + \mathfrak{r})(r + \mathfrak{r}^2)(\lambda' + \mathfrak{r}) = \lambda r \lambda' + \mathfrak{r}^2$, where $\lambda, \lambda' \in \Lambda$ and $r \in \mathfrak{r}$. Obviously $\mathfrak{r}/\mathfrak{r}^2 = (\Lambda/\mathfrak{r})(\mathfrak{r}/\mathfrak{r}^2)(\Lambda/\mathfrak{r})$. By using the decomposition $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} D_i$, we get a decomposition $\mathfrak{r}/\mathfrak{r}^2 \simeq \bigoplus_{i,j \in I} D_j \mathfrak{r}/\mathfrak{r}^2 D_i = \bigoplus_{i,j \in I} ({}_j M_i)$.

Let V be a $\Sigma - \Sigma$ -bimodule, where Σ is a ring. We write $V^{(n)}$ for the n -fold tensor product $V \otimes_\Sigma V \otimes_\Sigma \dots \otimes_\Sigma V$, and we let $V^{(0)} = \Sigma$. The *tensor ring* of Σ and V is defined as the graded ring $T(\Sigma, V) = V^{(0)} \oplus V^{(1)} \oplus V^{(2)} \oplus \dots$, where multiplication $V^{(n)} \times V^{(m)} \rightarrow V^{(n+m)}$ is given using the tensor product over Σ : For $\sum_{i \in I} a_{i,1} \otimes \dots \otimes a_{i,n} \in V^{(n)}$ and $\sum_{j \in J} b_{j,1} \otimes \dots \otimes b_{j,m} \in V^{(m)}$ we let $(\sum_{i \in I} a_{i,1} \otimes \dots \otimes a_{i,n})(\sum_{j \in J} b_{j,1} \otimes \dots \otimes b_{j,m}) = \sum_{i \in I, j \in J} a_{i,1} \otimes \dots \otimes a_{i,n} \otimes b_{j,1} \otimes \dots \otimes b_{j,m}$. If Σ is a k -algebra and k acts centrally on V , then k acts centrally on all $V^{(n)}$, and we can view $T(\Sigma, V)$ as a k -algebra by letting $l(\sum_{i \in I} a_{i,1} \otimes \dots \otimes a_{i,n}) = \sum_{i \in I} l a_{i,1} \otimes \dots \otimes a_{i,n}$ for $l \in k$ and $\sum_{i \in I} a_{i,1} \otimes \dots \otimes a_{i,n} \in V^{(n)}$, where the multiplication $l a_{i,1}$ is taken using the k -algebra structure of V . When we view the tensor ring $T(\Sigma, V)$ as a k -algebra, we call it the *tensor algebra* of Σ and V .

A morphism between two k -algebras is called a *k -algebra homomorphism* if it is a ring homomorphism when the algebras are viewed as rings, and at the same time a k -homomorphism when the algebras are viewed as k -modules.

We say that a k -algebra Λ *splits* or that Λ is a *split algebra* if the natural projection $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$ splits in the sense that there exists a k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon = \text{id}_{\Lambda/\mathfrak{r}}$. Observe that ϵ is not unique. By the Wedderburn-Malcev theorem Λ splits when $\sup\{n \mid H_R^n(\Lambda, M) \neq (0) \text{ for some } \Lambda\text{-bimodule } M\} \leq 1$, where $H_R^n(\Lambda, M)$ is the n 'th Hochschild cohomology module of Λ with coefficients in M [Pi, p. 209]. This happens in particular when k is a perfect field, as we will see in Proposition 3.10.

We end this section with an outline of the proof of Gabriel's structure theorem, which says that an elementary (or equivalently basic) finite dimensional algebra Λ over an algebraically closed field k is isomorphic to a quotient of the path algebra $k\Gamma$ of a finite quiver Γ [ARS, Theorem III.1.9]. Take the tensor algebra T of the k -algebra Λ/\mathfrak{r} and the $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule $\mathfrak{r}/\mathfrak{r}^2$. This is a k -algebra, and there exists a k -algebra epimorphism $\tilde{f}: T \rightarrow \Lambda$. The tensor algebra T is isomorphic to the path algebra $k\Gamma$ of a finite quiver Γ , so we get a k -algebra epimorphism $k\Gamma \simeq T \rightarrow \Lambda$, which shows that Λ is isomorphic to a quotient of a path algebra.

2. SPECIES

In this section we want to introduce the notion of *species with relations*. Similar ideas have been used before (e.g. in [As] under the name *bounden species*), but then only as an ideal in the algebra corresponding to the species. In contrast, we want to introduce relations for a species in a similar fashion as was done for relations in the path algebra. Throughout this article we will work with left modules, there are dual definitions and proofs in the right module case.

A *species* (also known as a *modulated quiver*) $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ is a set of division rings D_i and $D_j - D_i$ -bimodules ${}_jM_i$ such that $\text{Hom}_{D_i}({}_jM_i, D_i) \simeq \text{Hom}_{D_j}({}_jM_i, D_j)$ as $D_i - D_j$ -bimodules. We say a species $(D_i, {}_jM_i)_{i,j \in I}$ is a *k-species* if all the division rings D_i are finite dimensional over a common central subfield k , all the bimodules ${}_jM_i$ are finite dimensional over k , and $\lambda m = m\lambda$ for all $\lambda \in k$ and $m \in {}_jM_i$. So for a *k-species* we have $\text{Hom}_{D_i}({}_jM_i, D_i) \simeq \text{Hom}_{D_j}({}_jM_i, D_j)$ as $D_i - D_j$ -bimodules, hence our definition of *k-species* is similar to the definition of a *k-species* given in [Ga1] and [Ri]. All species we will work with in connection with finite dimensional *k*-algebras are *k-species*.

To visualize a species $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ we draw a quiver where we use the division rings D_i as vertices, and for each non-zero bimodule ${}_jM_i$ we draw an arrow starting in D_i and ending in D_j and index the arrow using the bimodule ${}_jM_i$. For example the species given by the division rings $D_1 = \mathbb{R}$ and $D_2 = \mathbb{C}$, and the bimodules ${}_2M_1 = \mathbb{C}$ and ${}_1M_2 = (0)$, will be drawn as

$$\mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C}$$

We will call the division rings D_i the vertices of \mathcal{S} , and when we view D_i as a vertex we will sometimes just call it i .

The *underlying quiver* $Q_{\mathcal{S}}$ of \mathcal{S} is the quiver with vertices $i \in I$ and arrows $i \rightarrow j$ for all ${}_jM_i \neq (0)$. We say that a species \mathcal{S} is finite if the underlying quiver $Q_{\mathcal{S}}$ is finite, and we say that \mathcal{S} is without oriented cycles if there are no oriented cycles $i \rightarrow i_1 \rightarrow \dots \rightarrow i_{n-1} \rightarrow i$ in $Q_{\mathcal{S}}$.

A *representation* $V = (V_i, {}_j\phi_i)$ over a species $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ is a set of left D_i -modules V_i together with morphisms

$${}_j\phi_i: {}_jM_i \otimes_{D_i} V_i \rightarrow V_j$$

where ${}_jM_i \otimes_{D_i} V_i$ is viewed as a left D_j -module. Composition of morphisms ${}_k\phi_j \circ {}_j\phi_i = {}_k\phi'_i: ({}_kM_j \otimes_{D_j} {}_jM_i) \otimes_{D_i} V_i \rightarrow V_k$ is given by ${}_k\phi'_i(({}_k m_j \otimes_{D_j} {}_j m_i)) \otimes_{D_i} v_i = {}_k\phi_j({}_k m_j \otimes_{D_j} ({}_j\phi_i({}_j m_i \otimes_{D_i} v_i)))$, where ${}_k m_j \in {}_kM_j$, ${}_j m_i \in {}_jM_i$ and $v_i \in V_i$.

Let ${}_j\mathcal{P}_i$ be the set of all paths p in $Q_{\mathcal{S}}$ which start in the vertex i and end in j , and let n_p be the length of the path p . The vertices in the path p will be denoted $p(l)$ for $0 \leq l \leq n_p$ in such a way that p is the path $i = p(0) \rightarrow p(1) \rightarrow \dots \rightarrow p(n_p - 1) \rightarrow p(n_p) = j$. We then have a $D_j - D_i$ bimodule

$${}_j\mathcal{M}_i = \bigoplus_{p \in {}_j\mathcal{P}_i} {}_jM_{p(n_p-1)} \otimes_{D_{p(n_p-1)}} \dots \otimes_{D_{p(1)}} {}_{p(1)}M_i$$

Using the composing of morphisms described above, from the morphisms $\{{}_j\phi_i\}_{i,j \in I}$ we induce a unique morphism ${}_j f_i: {}_j\mathcal{M}_i \otimes_{D_i} V_i \rightarrow V_j$ for each pair $i, j \in I$.

The set of representations $V = (V_i, {}_j\phi_i)$ over a species $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ gives rise to an abelian category $\text{Rep } \mathcal{S}$ in which a morphism $\alpha: V = (V_i, {}_j\phi_i) \rightarrow$

$(V'_i, {}_j\phi'_i) = V'$ is a set of D_i -linear maps $\alpha_i: V_i \rightarrow V'_i$ such that the diagram

$$\begin{array}{ccc} {}_jM_i \otimes_{D_i} V_i & \xrightarrow{1 \otimes \alpha_i} & {}_jM_i \otimes_{D_i} V'_i \\ \downarrow {}_j\phi_i & & \downarrow {}_j\phi'_i \\ V_j & \xrightarrow{\alpha_j} & V'_j \end{array}$$

commutes for all $i, j \in I$. The full abelian subcategory of $\text{Rep } \mathcal{S}$ consisting of all representations V for which all V_i are finite dimensional as vector spaces over k will be denoted $\text{rep } \mathcal{S}$.

For a species $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ we let the *tensor algebra* $T(\mathcal{S})$ of \mathcal{S} be the tensor ring $T(D, M) = T(\oplus_{i \in I} D_i, \oplus_{i,j \in I} {}_jM_i)$. Here we view $M = \oplus_{i,j \in I} {}_jM_i$ as a $D = \oplus_{i \in I} D_i$ bimodule the natural way. Since D is a k -algebra and k acts centrally on M , we know that $T(\mathcal{S})$ is a k -algebra. Let J denote the ideal $\oplus_{i \geq 1} M^{(i)}$ in $T(\mathcal{S})$. Then $T(\mathcal{S})/J \simeq D = \oplus_{i \in I} D_i$ is semisimple.

The relation between representations over a species and the modules over the corresponding tensor algebra is similar to the correspondence between quiver representations and modules over the corresponding tensor algebra, as shown in the following proposition. Although this next proposition is known, we still include the proof here since ideas from it will be used repeatedly in the rest of this article.

Proposition 2.1. [DR1, Proposition 10.1] *Let $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ be a finite k -species. Then the category $\text{Rep } \mathcal{S}$ and the category $\text{Mod } T(\mathcal{S})$ of left $T(\mathcal{S})$ -modules are equivalent.*

Proof. We want to define two functors

$$F: \text{Rep } \mathcal{S} \rightleftarrows \text{Mod } T(\mathcal{S}): G$$

such that $G \circ F \simeq \text{id}_{\text{Rep } \mathcal{S}}$ and $F \circ G \simeq \text{id}_{\text{Mod } T(\mathcal{S})}$.

We start with G . Let $V \in \text{Mod } T(\mathcal{S})$. Since $D = \oplus_{i \in I} D_i$ is a subring of $T(\mathcal{S})$, we can view V as a left D -module. Since $1_{T(\mathcal{S})} \in D$ we have $DV = V$, therefore $V = (\oplus_{i \in I} D_i)V = \oplus_{i \in I} (D_i V) = \oplus_{i \in I} V_i$ where $V_i = D_i V$, hence the central idempotents in D decompose V as a left D -module.

View $M = \oplus_{i,j \in I} ({}_jM_i)$ as a $D - D$ -bimodule, then ${}_jM_i D_l = (0)$ for $l \neq i$. Since V is a $T(\mathcal{S})$ -module and M is a subset of $T(\mathcal{S})$, we get a morphism $\phi': M \times V \rightarrow V$ where $\phi'(m, v) = mv$ by using the $T(\mathcal{S})$ -module structure on V . This morphism is D -biadditive, so it gives rise to an additive morphism $\phi: M \otimes_D V \rightarrow V$ where $\phi(m \otimes v) = mv$. We view ϕ as a left D -module morphism using that M is a left D -module. Since ${}_jM_k \otimes_D V_l = {}_jM_k \otimes_D (D_l V) = (0)$ for all $k \neq l$, we see that $M \otimes_D V \simeq \oplus_{i,j \in I} ({}_jM_i \otimes_{D_i} V_i)$. Observe that $\phi({}_jM_i \otimes_{D_i} V_i) = \phi(D({}_jM_i \otimes_{D_i} V_i)) = \phi(D_j({}_jM_i \otimes_{D_i} V_i)) = D_j \phi({}_jM_i \otimes_{D_i} V_i) \subseteq D_j V = V_j$. Let ${}_j\phi_i = \phi|_{{}_jM_i \otimes_{D_i} V_i}: {}_jM_i \otimes_{D_i} V_i \rightarrow V_j$. Define G on objects by letting $G(V) = (V_i, {}_j\phi_i)_{i,j \in I}$.

Let $\alpha: V \rightarrow V' \in \text{Mod } T(\mathcal{S})$. Since α is a morphism of left $T(\mathcal{S})$ -modules, it is also a morphism of left $D = \oplus_{i \in I} D_i$ -modules, and then in particular a left D_i -module morphism for every $i \in I$, hence $\alpha(V_i) \subseteq V'_i$. Let $\alpha_i = \alpha|_{V_i}: V_i \rightarrow V'_i$, and let $G(\alpha) = \{\alpha_i\}_{i \in I}$. To see that $\{\alpha_i\}_{i \in I}$ is a map of \mathcal{S} representations, we need to check that ${}_j\phi'_i \circ (1 \otimes \alpha_i) = \alpha_j \circ {}_j\phi_i$ for all $i, j \in I$. Using that α is a morphism of $T(\mathcal{S})$ -modules, we have

$$\begin{aligned} {}_j\phi'_i \circ (1 \otimes \alpha_i)(m \otimes v) &= {}_j\phi'_i(m \otimes \alpha_i(v)) = m\alpha_i(v) = m\alpha(v) \\ &= \alpha(mv) = \alpha_j(mv) = \alpha_j \circ {}_j\phi_i(m \otimes v) \end{aligned}$$

for $m \in {}_jM_i$ and $v \in V_i$. This shows that $G: \text{Mod } T(\mathcal{S}) \rightarrow \text{Rep } \mathcal{S}$ is a functor.

We then have to construct a functor $F: \text{Rep } \mathcal{S} \rightarrow \text{Mod } T(\mathcal{S})$. For an object $(V_i, {}_j\phi_i)_{i,j \in I}$ in $\text{Rep } \mathcal{S}$, let $V = \bigoplus_{i \in I} V_i$. Let D operate on V the obvious way, namely using the left D_i structure on V_i , and letting $D_j V_i = (0)$ for all $j \neq i$. View ${}_j\phi_i$ as a morphism $M \otimes_D V \rightarrow V$ by letting ${}_j\phi_i|_{M_k \otimes_{D_k} V_k} = 0$ for $k \neq i$ and $l \neq j$ where $k, i, l, j \in I$. Then we can define $\phi = \sum_{i,j \in I} {}_j\phi_i: M \otimes_D V \rightarrow V$. Let M operate on V using ϕ , hence for $m \in M$ and $v \in V$, let $mv = \phi(m \otimes v)$. By induction we define $M^{(n)} \times V = M \otimes_D \cdots \otimes_D M \times V \rightarrow V$ using the morphism $\phi^{(n)}: M^{(n)} \otimes_D V \rightarrow V$ where $\phi^{(n)} = \phi(1 \otimes_D \phi^{(n-1)})$ and $\phi^{(1)} = \phi$. Let $F((V_i, {}_j\phi_i)_{i,j \in I}) = V$ where V has this $T(\mathcal{S})$ -module structure.

Let $\{\alpha_i\}_{i \in I}: (V_i, {}_j\phi_i)_{i,j \in I} \rightarrow (V'_i, {}_j\phi'_i)_{i,j \in I}$ be a morphism in $\text{Rep } \mathcal{S}$. View α_i as a morphism on V by letting $\alpha_i|_{V_j} = 0$ for $j \neq i \in I$, and let $\alpha = \sum_{i \in I} \alpha_i: V = \bigoplus_{i \in I} V_i \rightarrow \bigoplus_{i \in I} V'_i = V'$. Since all α_i are left D_i -linear, we only need to show that $\alpha(mv) = m\alpha(v)$ for $m \in \bigoplus_{i \geq 1} M^{(i)}$. Since multiplication by an element in $\bigoplus_{i \geq 1} M^{(i)}$ is induced by the multiplication of elements in M , this is true if $\alpha(mv) = m\alpha(v)$ for $m \in M$. Invoking that α is D -linear and $M = \bigoplus_{i,j \in I} {}_jM_i$ as a $D - D$ -bimodule, what one needs to show is that $\alpha(mv) = m\alpha(v)$ for $m \in {}_jM_i$. Using that $\{\alpha_i\}_{i \in I}$ is a morphism in $\text{Rep } \mathcal{S}$ we see that $\alpha(mv) = \alpha_j({}_j\phi_i(m \otimes v)) = {}_j\phi'_i \circ (1 \otimes \alpha_i)(m \otimes v) = {}_j\phi'_i(m \otimes \alpha_i(v)) = m\alpha(v)$. This shows that $F: \text{Rep } \mathcal{S} \rightarrow \text{Mod } T(\mathcal{S})$ is a functor.

Observing that $G \circ F \simeq \text{id}_{\text{Rep } \mathcal{S}}$ and $F \circ G \simeq \text{id}_{\text{Mod } T(\mathcal{S})}$, we have proven the proposition. \square

Corollary 2.2. *Let $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ be a finite k -species. Then the category $\text{rep } \mathcal{S}$ and the category $\text{mod } T(\mathcal{S})$ of finite dimensional left $T(\mathcal{S})$ -modules are equivalent.*

Proof. From Proposition 2.1 we have two functors

$$F: \text{Rep } \mathcal{S} \rightleftarrows \text{Mod } T(\mathcal{S}): G$$

such that $G \circ F \simeq \text{id}_{\text{Rep } \mathcal{S}}$ and $F \circ G \simeq \text{id}_{\text{Mod } T(\mathcal{S})}$. We want to show that $F|_{\text{rep } \mathcal{S}} \subset \text{mod } T(\mathcal{S})$ and $G|_{\text{mod } T(\mathcal{S})} \subset \text{rep } \mathcal{S}$.

Therefore let $(V_i, {}_j\phi_i) \in \text{rep } \mathcal{S}$. Then $\dim_k V_i < \infty$, and since \mathcal{S} is finite we get that $V = \bigoplus_{i \in I} V_i$ is finite dimensional over k too. Hence $F((V_i, {}_j\phi_i)) = V \in \text{mod } T(\mathcal{S})$.

Now let $V \in \text{mod } T(\mathcal{S})$. Then $\dim_k V < \infty$, therefore $V_i = D_i V$ is finite dimensional over k too, so $G(V) = (V_i, {}_j\phi_i) \in \text{rep } \mathcal{S}$. \square

A relation σ of a species $\mathcal{S} = (D_i, {}_jM_i)_{i,j \in I}$ is a sum $\sigma = g_1 + \cdots + g_n$ of elements $g_l = g_{l,n_l} \otimes \cdots \otimes g_{l,1} \in {}_{i(n_l,l)}M_{i(n_l-1,l)} \otimes {}_{D_{i(n_l-1,l)}} \cdots \otimes {}_{D_{i(l,1)}} {}_{i(l,1)}M_{i(l,0)}$ where $i(l,n_l) = b$ and $i(l,0) = a$ for all $1 \leq l \leq n$. We will write the relation σ as ${}_b\sigma_a$ when we want to emphasize that it starts in a and ends in b . Let $\rho = \{\sigma_t\}_{t \in T}$ be a set of relations, where the different elements σ_t possibly start and end in different vertices. We call the pair (\mathcal{S}, ρ) a *species with relations*. Define $T(\mathcal{S}, \rho) = T(\mathcal{S})/\langle \rho \rangle$ where $\langle \rho \rangle$ is the ideal in $T(\mathcal{S})$ generated by the elements $\{\sigma_t\}_{t \in T}$. Also, define $\text{Rep}(\mathcal{S}, \rho)$ as the category of representations $V \in \text{Rep } \mathcal{S}$ for which ${}_j f_i|_{\langle {}_j\sigma_i \rangle \otimes_{D_i} V_i} = 0$ whenever there is an element ${}_j\sigma_i \in \rho$, where $\langle {}_j\sigma_i \rangle$ is the subspace of ${}_jM_i$ generated by ${}_j\sigma_i$ as a $D_j - D_i$ -bimodule. Let $\text{rep}(\mathcal{S}, \rho) = \text{Rep}(\mathcal{S}, \rho) \cap \text{rep } \mathcal{S}$.

The next proposition is a generalization of [ARS, Proposition 1.7].

Proposition 2.3. *Let \mathcal{S} be a finite k -species, and ρ a set of relations. Then the category $\text{Rep}(\mathcal{S}, \rho)$ and the category $\text{Mod}(T(\mathcal{S})/\langle \rho \rangle)$ of left $T(\mathcal{S})/\langle \rho \rangle$ -modules are equivalent.*

Proof. Recall from Proposition 2.1 that we have mutually inverse equivalences

$$F: \text{Rep } \mathcal{S} \rightleftarrows \text{Mod } T(\mathcal{S}): G$$

We want to show that these functors induce an equivalence between $\text{Rep}(\mathcal{S}, \rho)$ and $\text{Mod}(T(\mathcal{S})/\langle \rho \rangle)$.

Let $(V_i, {}_j\phi_i)_{i,j \in I} \in \text{Rep}(\mathcal{S}, \rho)$. Since ${}_j f_i \mid_{\langle {}_j\sigma_i \rangle \otimes_{D_i} V_i} = 0$ for every element ${}_j\sigma_i \in \rho$, we have $\langle {}_j\sigma_i \rangle F((V_i, {}_j\phi_i)_{i,j \in I}) = (0)$, where in the last equation $\langle {}_j\sigma_i \rangle$ is the ideal in $T(\mathcal{S})$ generated by ${}_j\sigma_i$. Therefore $\langle \rho \rangle F((V_i, {}_j\phi_i)_{i,j \in I}) = (0)$, so $F((V_i, {}_j\phi_i)_{i,j \in I}) \in \text{Mod}(T(\mathcal{S})/\langle \rho \rangle)$.

On the other hand, let $V \in \text{Mod}(T(\mathcal{S})/\langle \rho \rangle)$. For ${}_j\sigma_i \in \rho$, we have that $(0) = \langle {}_j\sigma_i \rangle V = \langle {}_j\sigma_i \rangle V_i$, which implies that ${}_j f_i \mid_{\langle {}_j\sigma_i \rangle \otimes_{D_i} V_i} = 0$ for the morphism ${}_j f_i: {}_j\mathcal{M}_i \otimes V_i \rightarrow V_j$ in $G(V)$, where $\langle {}_j\sigma_i \rangle$ in the last equation is the subset of ${}_j\mathcal{M}_i$ generated by ${}_j\sigma_i$ as a $D_j - D_i$ -bimodule. Hence $G(V) \in \text{Rep}(\mathcal{S}, \rho)$. \square

Corollary 2.4. *Let \mathcal{S} be a finite k -species, and ρ a set of relations. Then the category $\text{rep}(\mathcal{S}, \rho)$ and the category $\text{mod}(T(\mathcal{S})/\langle \rho \rangle)$ of finite dimensional left $T(\mathcal{S})/\langle \rho \rangle$ -modules are equivalent.*

Proof. This follows from Corollary 2.2 and Proposition 2.3. \square

Let $\Lambda = (T(\mathcal{S})/\langle \rho \rangle)$ where $\mathcal{S} = (D_i, {}_j\mathcal{M}_i)_{i,j \in I}$. In the category of left Λ -modules, the projective modules are $P_i = \Lambda D_i$ for $i \in I$, where $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} D_i$. Observe that there is a one-to-one correspondence between the vertices I and the indecomposable projective representations of (\mathcal{S}, ρ) .

3. BASIC ALGEBRAS OVER PERFECT FIELDS

In this section, let Λ be a finite dimensional basic algebra over a field k (not necessarily algebraically closed). We want to investigate algebras Λ that are split, i.e. the natural projection onto the quotient algebra $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$ splits in the sense that there exists a k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon = \text{id}_{\Lambda/\mathfrak{r}}$.

Assume an algebra Λ is split. Recall that the k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon = \text{id}_{\Lambda/\mathfrak{r}}$ is not unique. Using $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ we can view $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} D_i$ as a subalgebra of Λ , and we can identify the division rings D_i with their image in Λ under ϵ . Observe that the subalgebra structure of Λ/\mathfrak{r} and the identification of D_i with a subset of Λ is dependent on the choice of ϵ .

We say that a split algebra Λ is \mathfrak{r} -split if for any k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon = \text{id}_{\Lambda/\mathfrak{r}}$, the short exact sequence

$$0 \rightarrow \mathfrak{r}^2 \rightarrow \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{r}^2 \rightarrow 0$$

splits when we view the sequence as a sequence of $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodules.

In this section we will give a structure theorem for finite dimensional \mathfrak{r} -split basic algebras. At the end of this section we will show that all finite dimensional basic k -algebras over perfect fields k are \mathfrak{r} -split, hence they give rise to a large class of examples of \mathfrak{r} -split algebras.

To reach this goal we need a slight reformulation of [ARS, Lemma III.1.2]:

Lemma 3.1. *Let Σ be a k -algebra and V a $\Sigma - \Sigma$ -bimodule. Let Λ be a k -algebra and $f: \Sigma \oplus V \rightarrow \Lambda$ a morphism such that $f|_{\Sigma}: \Sigma \rightarrow \Lambda$ is a k -algebra homomorphism and $f|_V: V \rightarrow \Lambda$ is a $\Sigma - \Sigma$ -bimodule morphism when Λ is viewed as a $\Sigma - \Sigma$ -bimodule via $f|_{\Sigma}$. Then there exists a unique k -algebra homomorphism $\tilde{f}: T(\Sigma, V) \rightarrow \Lambda$ such that $\tilde{f}|_{\Sigma \oplus V} = f$.*

Proof. Let $\phi: V \times V \rightarrow \Lambda$ be given by $\phi(v_1, v_2) = f(v_1)f(v_2)$. Then for $r \in \Sigma$ we have $\phi(v_1r, v_2) = f(v_1r)f(v_2) = f(v_1)rf(v_2) = f(v_1)f(rv_2) = \phi(v_1, rv_2)$ since $f|_V$ is a $\Sigma - \Sigma$ -bimodule morphism. Also since $f|_V$ is a $\Sigma - \Sigma$ -bimodule morphism we get that $\phi(v_1 + v_2, v_3) = \phi(v_1, v_3) + \phi(v_2, v_3)$ and $\phi(v_1, v_2 + v_3) = \phi(v_1, v_2) + \phi(v_1, v_3)$, so ϕ is a Σ -biadditive morphism. Hence we get an induced additive morphism $f_2: V \otimes_{\Sigma} V \rightarrow \Lambda$ where $f_2(v_1 \otimes v_2) = f(v_1)f(v_2)$. Using again that

$f|_V$ is a $\Sigma - \Sigma$ -bimodule morphism, we see that for $r \in \Sigma$ we have $f_2(rv_1 \otimes v_2) = f(rv_1)f(v_2) = rf(v_1)f(v_2) = rf_2(v_1 \otimes v_2)$ and $f_2(v_1 \otimes v_2r) = f(v_1)f(v_2r) = f(v_1)f(v_2)r = f_2(v_1 \otimes v_2)r$, hence f_2 is a $\Sigma - \Sigma$ -bimodule morphism. By induction we construct a $\Sigma - \Sigma$ -bimodule morphism $f_n: V^{(n)} \rightarrow \Lambda$ where $f_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1)f(v_2) \cdots f(v_n)$.

Denote an element in $T(\Sigma, V)$ by $\sum_{i=0}^{\infty} v_i$, where $v_i \in V^{(i)}$ for all $i \in \mathbb{N}$ and v_i is zero for all but a finite number of $i \in \mathbb{N}$. If we let $f_0 = f|_{\Sigma}$ and $f_1 = f|_V$, we can define $\tilde{f}: T(\Sigma, V) \rightarrow \Lambda$ by letting $\tilde{f}(\sum_{i=0}^{\infty} v_i) = \sum_{i=0}^{\infty} f_i(v_i)$. For two elements $v = \sum_{i=0}^{\infty} v_i$ and $w = \sum_{i=0}^{\infty} w_i$ in $T(\Sigma, V)$ we see that

$$\begin{aligned} \tilde{f}(v+w) &= \tilde{f}(\sum_{i=0}^{\infty} v_i + w_i) \\ &= \sum_{i=0}^{\infty} f_i(v_i + w_i) \\ &= \sum_{i=0}^{\infty} (f_i(v_i) + f_i(w_i)) \\ &= \tilde{f}(\sum_{i=0}^{\infty} v_i) + \tilde{f}(\sum_{i=0}^{\infty} w_i) = \tilde{f}(v) + \tilde{f}(w) \\ \tilde{f}(vw) &= \tilde{f}(\sum_{i=0}^{\infty} \sum_{j=0}^i v_j w_{i-j}) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i f_i(v_j w_{i-j}) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i f_j(v_j) f_{i-j}(w_{i-j}) \\ &= (\sum_{i=0}^{\infty} f_i(v_i)) (\sum_{i=0}^{\infty} f_i(w_i)) \\ &= \tilde{f}(\sum_{i=0}^{\infty} v_i) \tilde{f}(\sum_{i=0}^{\infty} w_i) = \tilde{f}(v) \tilde{f}(w) \end{aligned}$$

This shows that \tilde{f} is a ring homomorphism. It is easy to see that \tilde{f} is a k -module homomorphism, hence \tilde{f} is a k -algebra homomorphism. Since $\{\Sigma, V\}$ generates $T(\Sigma, V)$, the morphism \tilde{f} is unique. \square

Since Λ is finite dimensional over k we know that $\mathfrak{r}/\mathfrak{r}^2$ is finitely generated as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. We can therefore find elements $\{r_1, r_2, \dots, r_m\}$ in \mathfrak{r} such that their images $\{\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m\}$ in $\mathfrak{r}/\mathfrak{r}^2$ generate $\mathfrak{r}/\mathfrak{r}^2$ as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. We let $\text{rl}(\Lambda)$ denote the Lowey length (radical length) of Λ , i.e. the smallest number $n \in \mathbb{N}$ such that $\mathfrak{r}^n = (0)$.

The following result is a generalization of [ARS, Theorem III.1.9 (a)(b)]. The proof of part (a) follows the lines of [ARS, Theorem III.1.9 (a)], and can be found in [Li, Lemma 3.1 (i)]. We include the proof here for completeness. Part (b) could have been proven similarly to the proof of [ARS, Theorem III.1.9 (b)]. We will use another proof since ideas from it will be used later in Proposition 4.1.

Proposition 3.2. *Let Λ be a finite dimensional basic \mathfrak{r} -split k -algebra.*

- (a) *Let $\{r_1, r_2, \dots, r_m\}$ be elements in \mathfrak{r} such that their images $\{\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m\}$ in $\mathfrak{r}/\mathfrak{r}^2$ generate $\mathfrak{r}/\mathfrak{r}^2$ as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. Then $\{D_1, D_2, \dots, D_n, r_1, r_2, \dots, r_m\}$ generate Λ as a k -algebra, where $\Lambda/\mathfrak{r} \simeq \bigoplus_{i=1}^n D_i$ is viewed as a k -subalgebra of Λ using a k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon = \text{id}_{\Lambda/\mathfrak{r}}$.*
- (b) *There is a surjective k -algebra homomorphism $\tilde{f}: T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ such that $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)} \subset \ker \tilde{f} \subset \bigoplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$.*

Proof. (a) We will prove this by induction on the Lowey length of Λ . So assume $\text{rl}(\Lambda) = 1$. Then $\mathfrak{r} = (0)$, so $\Lambda = \bigoplus_{i=1}^n D_i$, and Λ is obviously generated as a k -algebra by the set $\{D_1, D_2, \dots, D_n\}$.

When $\text{rl}(\Lambda) = 2$, then $\mathfrak{r}^2 = (0)$. Since $\Lambda/\mathfrak{r} \simeq \bigoplus_{i=1}^n D_i$ we have $\Lambda = \langle D_1, \dots, D_n \rangle + \mathfrak{r}$. Moreover $\mathfrak{r} \simeq \mathfrak{r}/\mathfrak{r}^2$, so we see that Λ is generated by the set $\{D_1, \dots, D_n, r_1, \dots, r_m\}$.

Assume (a) is true for algebras with Lowey length m , and assume $\text{rl}(\Lambda) = m + 1$. Let A be the k -subalgebra of Λ generated by

$\{D_1, D_2, \dots, D_n, r_1, r_2, \dots, r_m\}$, and let $x \in \Lambda$. Since $\text{rl}(A/(A \cap \mathfrak{r}^m)) = \text{rl}(\Lambda/\mathfrak{r}^m) = m$, we have by induction that $A/(A \cap \mathfrak{r}^m) \simeq \Lambda/\mathfrak{r}^m$. Therefore there exists some $y \in A$ such that $x + \mathfrak{r}^m = y + (A \cap \mathfrak{r}^m)$. Then $x - y \in \mathfrak{r}^m$, hence $x - y = \sum_{i=1}^s \alpha_i \beta_i$ where $\alpha_i \in \mathfrak{r}^{m-1}$ and $\beta_i \in \mathfrak{r}$. Since $\mathfrak{r}^{m-1}/\mathfrak{r}^m \simeq (A \cap \mathfrak{r}^{m-1})/(A \cap \mathfrak{r}^m)$, we get $\alpha_i + \mathfrak{r}^m = a_i + (A \cap \mathfrak{r}^m)$ where $a_i \in A \cap \mathfrak{r}^{m-1}$. This yields $\alpha_i = a_i + a'_i$, where $a_i \in A \cap \mathfrak{r}^{m-1}$ and $a'_i \in \mathfrak{r}^m$. Similarly, since $\mathfrak{r}/\mathfrak{r}^m \simeq (A \cap \mathfrak{r})/(A \cap \mathfrak{r}^m)$, we get that $\beta_i + \mathfrak{r}^m = b_i + (A \cap \mathfrak{r}^m)$ where $b_i \in A \cap \mathfrak{r}$. Hence $\beta_i = b_i + b'_i$ where $b_i \in A \cap \mathfrak{r}$ and $b'_i \in \mathfrak{r}^m$. Then $\alpha_i \beta_i = (a_i + a'_i)(b_i + b'_i) = a_i b_i$ since $a_i b'_i \in \mathfrak{r}^{2m-1} = (0)$, $a'_i b_i \in \mathfrak{r}^{m+1} = (0)$, and $a'_i b'_i \in \mathfrak{r}^{2m} = (0)$. This shows that $x - y \in A$, and since $y \in A$ we get that $x \in A$. Hence $\Lambda = A$, so Λ is generated by the set $\{D_1, \dots, D_n, r_1, \dots, r_m\}$.

- (b) Let $f|_{\Lambda/\mathfrak{r}} = \epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ be a lifting of the natural projection $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$, and view Λ/\mathfrak{r} as a k -subalgebra of Λ via ϵ . Since Λ is \mathfrak{r} -split, the sequence

$$0 \rightarrow \mathfrak{r}^2 \rightarrow \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{r}^2 \rightarrow 0$$

splits as a sequence of $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodules. Let $f|_{\mathfrak{r}/\mathfrak{r}^2}: \mathfrak{r}/\mathfrak{r}^2 \rightarrow \Lambda$ be given by the splitting map $\mathfrak{r}/\mathfrak{r}^2 \hookrightarrow \mathfrak{r}$ composed with the inclusion $\mathfrak{r} \hookrightarrow \Lambda$ viewed as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule morphism via ϵ .

Using Lemma 3.1 we get induced a k -algebra homomorphism $\tilde{f}: T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$, and by (a) this morphism is surjective. Since $\tilde{f}|_{\Lambda/\mathfrak{r} \oplus \mathfrak{r}/\mathfrak{r}^2}$ is a monomorphism, and the image intersects trivially with \mathfrak{r}^2 , we see that $\ker \tilde{f} \subset \bigoplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$. On the other hand, $\tilde{f}((\mathfrak{r}/\mathfrak{r}^2)^{(j)}) \subset \mathfrak{r}^j$, and since $\mathfrak{r}^{\text{rl}(\Lambda)} = (0)$, we get that $\ker \tilde{f} \supset \bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$. \square

As mentioned, the result [ARS, Theorem III.1.9 (a)(b)] is a special case of Proposition 3.2. In [ARS, Theorem III.1.9 (a)(b)] it is assumed that Λ is a finite dimensional basic k -algebra where k is an algebraically closed field, and we will show in Corollary 3.11 that this implies that Λ is \mathfrak{r} -split. The main difference between Proposition 3.2 and [ARS, Theorem III.1.9 (a)(b)] is that we have replaced the complete set of pairwise orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ with the set $\{D_1, D_2, \dots, D_n\}$. The reason for this change is that in the algebraically closed case e_i would have generated D_i as a k -algebra, but in our non-algebraically closed case this is no longer true.

Let Λ be a finite dimensional basic k -algebra. Then $\Lambda/\mathfrak{r} = \bigoplus_{i \in I} D_i = D$, and

$$\begin{aligned} \mathfrak{r}/\mathfrak{r}^2 &= (\Lambda/\mathfrak{r})(\mathfrak{r}/\mathfrak{r}^2)(\Lambda/\mathfrak{r}) = (\bigoplus_{i \in I} D_i)(\mathfrak{r}/\mathfrak{r}^2)(\bigoplus_{i \in I} D_i) \\ &= \bigoplus_{i, j \in I} (D_j(\mathfrak{r}/\mathfrak{r}^2)D_i) = \bigoplus_{i, j \in I} ({}_j M_i) = M \end{aligned}$$

Since k sits inside the center of Λ , we have $k \subset D$. Moreover $\lambda \lambda' = \lambda' \lambda$ and $\lambda m = m \lambda$ for all $\lambda \in k$, $\lambda' \in D_i$ and $m \in {}_j M_i$ for all $i, j \in I$. Since Λ is finite dimensional, we know that D_i and ${}_j M_i$ are finite dimensional over k for all $i, j \in I$. Hence $S_\Lambda = (D_i, {}_j M_i)_{i, j \in I}$ is a k -species, and it will be called the *species* of Λ . Observe that we do not assume Λ is \mathfrak{r} -split to define the species of Λ .

Remember that we denote the ideal $\bigoplus_{i \geq 1} M^{(i)}$ in $T(\mathcal{S})$ by J , where \mathcal{S} is the species $(D_i, {}_j M_i)_{i, j \in I}$ and $M = \bigoplus_{i, j \in I} ({}_j M_i)$.

Proposition 3.3. *Let Λ be a finite dimensional basic \mathfrak{r} -split k -algebra. Then $\Lambda \simeq T(S_\Lambda)/\langle \rho \rangle$ with $J^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset J^2$, where S_Λ is the species of Λ and ρ is a set of relations.*

Proof. Observe that $T(S_\Lambda) = T(D, M) = T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)$. From Proposition 3.2 we have an epimorphism $\tilde{f}: T(S_\Lambda) = T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ with $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)} \subset \ker \tilde{f} \subset \bigoplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$. Let $\rho' = \{\sigma'_t\}_{t \in T'}$ be a set of generators for $\ker \tilde{f}$ as an ideal

in $T(\mathcal{S}_\Lambda)$. Since $\mathfrak{r}/\mathfrak{r}^2 = M$ and $\ker \tilde{f} = \langle \rho' \rangle$, we have $J^{\text{rl}(\Lambda)} = \bigoplus_{j \geq \text{rl}(\Lambda)} M^{(j)} \subset \langle \rho' \rangle \subset \bigoplus_{j \geq 2} M^{(j)} = J^2$.

We want to transfer ρ' into a set of relations. Let $\{e_i\}_{i \in I}$ be a complete set of pairwise orthogonal primitive idempotents in Λ . Let $\sigma'_t \in \rho$, then $\sigma'_t = 1_\Lambda \sigma'_t 1_\Lambda = (\sum_{i \in I} e_i) \sigma'_t (\sum_{i \in I} e_i) = \sum_{i,j \in I} e_j \sigma'_t e_i = \sum_{i,j \in I} ({}_j \rho_{ti})$ where ${}_j \rho_{ti} = e_j \sigma'_t e_i$. All ${}_j \rho_{ti}$ are sets of relations, so letting $\rho = \bigcup_{t \in T'} \bigcup_{i,j \in I} ({}_j \rho_{ti})$, we see that $\langle \rho \rangle = \langle \rho' \rangle$ where ρ is a set of relations.

Using $T(\mathcal{S}_\Lambda)/\langle \rho \rangle \simeq T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)/\ker \tilde{f} \simeq \Lambda$, we have proven the proposition. \square

Observe that the set of relations $\rho = \{\sigma_t\}_{t \in T}$ in Proposition 3.3 can be chosen to be finite: The $\text{rl}(\Lambda)$ -fold tensor product $M^{(\text{rl}(\Lambda))} = (\mathfrak{r}/\mathfrak{r}^2)^{(\text{rl}(\Lambda))}$ is finite dimensional over k , so there exists a finite set $\{\sigma_t\}_{t \in T'}$ of generators, and this set of generators can be chosen to consist of relations. Since $(\mathfrak{r}/\mathfrak{r}^2)^{(\text{rl}(\Lambda))}$ generates $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$, the finite set $\{\sigma_t\}_{t \in T'}$ also generates $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$. Since $\ker \tilde{f}/(\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)})$ is finite dimensional, there exists a finite set of elements $\{\sigma_t\}_{t \in T''}$ in $\ker \tilde{f}$ such that the corresponding elements in $\ker \tilde{f}/(\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)})$ is a generating set, and also $\{\sigma_t\}_{t \in T''}$ can be chosen to consist of relations. Then, letting $T = T' \cup T''$, we know $\{\sigma_t\}_{t \in T}$ to be a finite set of relations which generates $\ker \tilde{f}$.

Using Corollary 2.4 and Proposition 3.3 we get the following corollary.

Corollary 3.4. *Let Λ be a finite dimensional basic \mathfrak{r} -split k -algebra, and let \mathcal{S}_Λ be the species of Λ . Then the category $\text{mod } \Lambda$ of finite dimensional left Λ -modules is equivalent to $\text{rep}(\mathcal{S}_\Lambda, \rho)$ where ρ is a set of relations such that $J^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset J^2$.*

We now want to investigate finite dimensional *hereditary algebras*, i.e. finite dimensional algebras where all left ideals are projective. Hereditary algebras have been studied thoroughly, in particular the rest of the results in this section are either well known or similar to well known results. The next two lemmas are restated for completeness.

Lemma 3.5. [ARS, Lemma III.1.11] *If Λ is a basic finite dimensional hereditary algebra and \mathfrak{a} is a non-zero ideal of Λ contained in \mathfrak{r}^2 , then Λ/\mathfrak{a} is not hereditary.*

Lemma 3.6. [ARS, Lemma III.1.12] *If Λ is a basic finite dimensional hereditary algebra, and $f: P \rightarrow Q$ is a non-zero morphism between indecomposable projective Λ -modules, then f is a monomorphism.*

Proposition 3.7. *Let Λ be a basic finite dimensional hereditary \mathfrak{r} -split k -algebra, let \mathcal{S}_Λ be the species of Λ , and $Q_{\mathcal{S}_\Lambda}$ the underlying quiver of \mathcal{S}_Λ . Then $Q_{\mathcal{S}_\Lambda}$ is finite and without oriented cycles, and Λ is isomorphic to $T(\mathcal{S}_\Lambda)$.*

Proof. We know from Proposition 3.3 that $\Lambda \simeq T(\mathcal{S}_\Lambda)/\langle \rho \rangle$ with $\langle \rho \rangle \subset J^2$. Since Λ is hereditary, Lemma 3.5 implies that $\langle \rho \rangle = (0)$, hence $\Lambda \simeq T(\mathcal{S}_\Lambda)$. Since Λ is finite dimensional, the underlying quiver $Q_{\mathcal{S}_\Lambda}$ of \mathcal{S}_Λ must be finite.

Assume there is an oriented cycle in $Q_{\mathcal{S}_\Lambda}$. Using Lemma 3.6, this will give rise to a proper monomorphism from an indecomposable projective P into itself, which contradicts that the algebra Λ is finite dimensional. \square

From the proposition above we see that a basic finite dimensional hereditary \mathfrak{r} -split k -algebra is a tensor algebra. Let us prove the opposite direction.

Lemma 3.8. *Let \mathcal{S} be a k -species with underlying quiver $Q_{\mathcal{S}}$. If $Q_{\mathcal{S}}$ is a finite quiver without oriented cycles, then $T(\mathcal{S})$ is a hereditary finite dimensional basic k -algebra.*

Proof. We know that $T(\mathcal{S}) \simeq \bigoplus_{i \in I} T(\mathcal{S})D_i \simeq \bigoplus_{i \in I} P_i$ is a decomposition of $T(\mathcal{S})$ into a direct sum of indecomposable projectives, and that $\text{rad } T(\mathcal{S}) = \bigoplus_{i \in I} (\bigoplus_{j: M_i \neq 0} (\dim_{D_j} M_i) P_j)$. Hence the radical of $T(\mathcal{S})$ is projective, which implies that $T(\mathcal{S})$ is hereditary [La2, Theorem 2.35].

Since $Q_{\mathcal{S}}$ is finite and without oriented cycles, $T(\mathcal{S})$ is finite dimensional. \square

We summarize the previous two results in the following theorem.

Theorem 3.9. *Let Λ be a basic finite dimensional \mathfrak{r} -split k -algebra, then the following are equivalent:*

- (i) Λ is hereditary
- (ii) Λ is isomorphic to a tensor algebra $T(\mathcal{S})$ of a species \mathcal{S} where the underlying quiver $Q_{\mathcal{S}}$ is finite and without oriented cycles

We now proceed to show there is a large class of finite dimensional basic k -algebras which are \mathfrak{r} -split, namely finite dimensional basic k -algebras where k is a perfect field.

Proposition 3.10. *If k is a perfect field and Λ is a finite dimensional basic k -algebra, then Λ is \mathfrak{r} -split.*

Proof. When k is perfect, Λ is split [Pi, Corollary 11.6]. We therefore only need to show that the sequence

$$0 \rightarrow \mathfrak{r}^2 \rightarrow \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{r}^2 \rightarrow 0$$

splits as a sequence of $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodules via any morphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ for which $\pi\epsilon \simeq \text{id}_{\Lambda/\mathfrak{r}}$ where $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$ is the natural projection morphism. Using the decomposition $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} D_i$, this is equivalent to showing that the sequence splits as a sequence of $D_i - D_j$ -bimodules for every pair $i, j \in I$.

A $D_i - D_j$ -bimodule M is given by an operation $D_i \times M \times D_j \rightarrow M$. By duality we get an equivalent operation $\phi: D_i \times D_j^{\text{op}} \times M \rightarrow M$. If k acts centrally on M and D_j , we have $\phi(d_i r, d_j, m) = (d_i r) m d_j = d_i m (r d_j) = d_i m (d_j r) = \phi(d_i, r d_j, m)$. Hence the operation ϕ is k -biadditive on $D_i \times D_j^{\text{op}}$, so it gives rise to an additive morphism $D_i \otimes_k D_j^{\text{op}} \times M \rightarrow M$. This way we can view M as a $D_i \otimes_k D_j^{\text{op}}$ -module. Since k acts centrally on all the objects in the short exact sequence and also on D_i and D_j , the sequence of $D_i - D_j$ -bimodules can be viewed as a sequence of $D_i \otimes_k D_j^{\text{op}}$ -modules.

Since k is perfect and D_j^{op} is a finite extension of k , we know that $D_i \otimes_k D_j^{\text{op}}$ is a semisimple k -algebra [DK, Theorem 5.3.6]. Therefore $D_i \otimes_k D_j^{\text{op}}$ -modules are projective [Ro, Theorem 4.13], so the sequence splits as a sequence of $D_i \otimes_k D_j^{\text{op}}$ -modules, hence it splits as a sequence of $D_i - D_j$ -modules. \square

Corollary 3.11. *If k is an algebraically closed field, and Λ is a finite dimensional basic k -algebra, then Λ is \mathfrak{r} -split.*

Proof. A field that is algebraically closed is in particular perfect, so this is a direct consequence of Lemma 3.10. \square

We will now summarize what we know about basic finite dimensional algebras over perfect fields.

Theorem 3.12. *Let Λ be a finite dimensional basic k -algebra where k is a perfect field, let \mathcal{S}_{Λ} be the species of Λ , and let $Q_{\mathcal{S}_{\Lambda}}$ be the underlying quiver of \mathcal{S}_{Λ} . Then the following holds:*

- (a) Let $\{r_1, r_2, \dots, r_m\}$ be elements in \mathfrak{r} such that their images $\{\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m\}$ in $\mathfrak{r}/\mathfrak{r}^2$ generate $\mathfrak{r}/\mathfrak{r}^2$ as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. Then $\{D_1, D_2, \dots, D_n, r_1, r_2, \dots, r_m\}$ generate Λ as a k -algebra, where $\Lambda/\mathfrak{r} \simeq \bigoplus_{i=1}^n D_i$.

- (b) *There is a surjective k -algebra homomorphism $\tilde{f}: T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ such that $\oplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^{(j)} \subset \ker \tilde{f} \subset \oplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^{(j)}$.*
- (c) *$\Lambda \simeq T(\mathcal{S}_\Lambda)/\langle \rho \rangle$ with $J^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset J^2$, where ρ is a set of relations on \mathcal{S}_Λ .*
- (d) *The category $\text{mod } \Lambda$ is equivalent to a category $\text{rep}(\mathcal{S}_\Lambda, \rho)$, where ρ is a set of relations on \mathcal{S}_Λ such that $J^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset J^2$.*
- (e) *Λ is hereditary if and only if $Q_{\mathcal{S}_\Lambda}$ is finite without oriented cycles and Λ is isomorphic to $T(\mathcal{S}_\Lambda)$.*

Example 3.13. As we know, not all finite dimensional algebras are split, and we will now give an example of such an algebra built on an example in [Ben, p. 99]. Let \mathbb{F}_2 be the Galois field consisting of two elements, and let $k = \mathbb{F}_2(x) = \{\frac{f}{g} \mid f, g \in \mathbb{F}_2[x], g \neq 0\}$ be the field of rational functions over \mathbb{F}_2 with indeterminate x . Observe that k is not a perfect field. We want to investigate the ring $\Lambda = k[y, z]/(z^2, y^2 - x - z)$. Note that this is a finite dimensional algebra over k , actually $\dim_k \Lambda = 4$, where $\{1, y, z, yz\}$ is a basis for Λ as a k -algebra. The Jacobson radical \mathfrak{r} of Λ is the ideal $\langle z \rangle$. This shows $\Lambda/\mathfrak{r} \simeq k[y]/(y^2 - x) \simeq \mathbb{F}_2(t)$ where in the last ring we have $t^2 = x$. Now $\mathbb{F}_2(t)$ is a finite field extension of k , and $\dim_k \mathbb{F}_2(t) = 2$ when we view $\mathbb{F}_2(t)$ as a k -algebra.

We want to check if Λ splits, hence we want to try to construct a k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$. Since Λ/\mathfrak{r} as a k -algebra is generated by $\{1, y\}$, we only need to define ϵ on the element y . To get a morphism we need $x = \epsilon(x) = \epsilon(y^2) = \epsilon(y)\epsilon(y)$. Since there are no solutions to the equation $u^2 - x$ for an indeterminate u in the ring Λ , this is impossible. Hence there are no k -algebra homomorphisms $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$, so Λ does not split. \triangle

4. BASIC FINITE DIMENSIONAL SPLIT ALGEBRAS

In this section we want to give a structure theorem for finite dimensional basic split k -algebras. The proof of this result will have many similarities with the proof for the structure theorem for finite dimensional basic \mathfrak{r} -split k -algebras (Proposition 3.3), but there are at the same time important differences. Even though \mathfrak{r} -split algebras in particular are split algebras, the structure theorem we are about to give for split algebras is not a generalization of the structure theorem for \mathfrak{r} -split algebras. On the other hand, it is a generalization of Gabriel's structure theorem for finite dimensional basic k -algebras where k is algebraically closed.

In contrast to the case when Λ was \mathfrak{r} -split, it is in general no longer true that the sequence

$$0 \rightarrow \mathfrak{r}^2 \rightarrow \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{r}^2 \rightarrow 0$$

splits when viewed as a sequence of $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodules via any k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon \simeq \text{id}_{\Lambda/\mathfrak{r}}$ for the natural projection $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$ (we know that such a k -algebra homomorphism ϵ exists since Λ is assumed to be split). The splitting of the above sequence is used in the construction of a tensor algebra mapping onto Λ . We will use a similar construction in this section, but we will only need a free $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule F such that F maps onto $\mathfrak{r}/\mathfrak{r}^2$. Using that F is free we get a lifting as shown in the following commutative diagram.

$$\begin{array}{ccccccc} & & & & F & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{r}^2 & \longrightarrow & \mathfrak{r} & \longrightarrow & \mathfrak{r}/\mathfrak{r}^2 \longrightarrow 0 \end{array}$$

We will replace the splitting morphism $\mathfrak{r}/\mathfrak{r}^2 \rightarrow \mathfrak{r}$ with the morphism $F \rightarrow \mathfrak{r}$ given by this lifting, we can use an argument similar to the one in the \mathfrak{r} -split case to construct a tensor algebra mapping onto Λ .

One natural choice for a free $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule mapping onto $\mathfrak{r}/\mathfrak{r}^2$ could have been $\Lambda/\mathfrak{r} \otimes_k \mathfrak{r}/\mathfrak{r}^2 \otimes_k \Lambda/\mathfrak{r}$, but this module will turn out to be unnecessarily large. Since we have $\Lambda/\mathfrak{r} \simeq \oplus D_i$ we only need to find a free $D_j - D_i$ -module ${}_jF_i$ mapping onto $D_j(\mathfrak{r}/\mathfrak{r}^2)D_i = {}_jM_i$ for all $i, j \in I$. Using that ${}_jF_i$ is free, we will get a lifting resulting in the following commutative diagram of $D_j - D_i$ -modules.

$$\begin{array}{ccccccc} & & & & {}_jF_i & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & D_j(\mathfrak{r}^2)D_i & \longrightarrow & D_j(\mathfrak{r})D_i & \longrightarrow & D_j(\mathfrak{r}/\mathfrak{r}^2)D_i \longrightarrow 0 \end{array}$$

A natural choice for such a module is ${}_j\tilde{M}_i = D_j \otimes_k D_j(\mathfrak{r}/\mathfrak{r}^2)D_i \otimes_k D_i = D_j \otimes_k ({}_jM_i) \otimes_k D_i$, where $\mathfrak{r}/\mathfrak{r}^2 \simeq \oplus_{i,j \in I} ({}_jM_i)$. An onto $D_j - D_i$ -module morphism ${}_jg_i: {}_j\tilde{M}_i \rightarrow {}_jM_i$ is given by ${}_jg_i(d_j \otimes {}_j m_i \otimes d_i) = d_j({}_j m_i)d_i$. Taking the direct sum we get the $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule $\tilde{M} = \oplus_{i,j \in I} ({}_j\tilde{M}_i)$. Let $g: \tilde{M} \rightarrow \mathfrak{r}/\mathfrak{r}^2$ be given by $g = \sum_{i,j \in I} ({}_jg_i)$ where we extend the functions ${}_jg_i$ to \tilde{M} by letting ${}_jg_i|_{\tilde{M}_k} = 0$ for $k \neq i$ and $l \neq j$ in I . Then g maps onto $\mathfrak{r}/\mathfrak{r}^2$ and factors through \mathfrak{r} , as shown in the diagram below.

$$\begin{array}{ccccccc} & & & & \tilde{M} & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{r}^2 & \longrightarrow & \mathfrak{r} & \longrightarrow & \mathfrak{r}/\mathfrak{r}^2 \longrightarrow 0 \end{array}$$

We define the *enlarged species* of Λ to be the species $\tilde{\mathcal{S}}_\Lambda = (D_i, {}_j\tilde{M}_i)_{i,j \in I}$. Since \tilde{M} is a Λ/\mathfrak{r} -bimodule we are able to define the *enlarged tensor algebra* of Λ as $\tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) = T(\Lambda/\mathfrak{r}, \tilde{M})$. The tensor algebra of the species $\tilde{\mathcal{S}}_\Lambda$ will then be the same as the enlarged tensor algebra of Λ since $T(\tilde{\mathcal{S}}_\Lambda) = T(\oplus_{i \in I} D_i, \oplus_{i,j \in I} ({}_j\tilde{M}_i)) = T(\Lambda/\mathfrak{r}, \tilde{M}) = \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)$. The n -fold tensor product $\tilde{M}^{(n)}$ can be simplified the following way:

$$\begin{aligned} \tilde{M}^{(n)} &= \tilde{M} \otimes_{\Lambda/\mathfrak{r}} \cdots \otimes_{\Lambda/\mathfrak{r}} \tilde{M} \\ &= (\oplus_{i,j \in I} D_i \otimes_k {}_iM_j \otimes_k D_j) \otimes_{\Lambda/\mathfrak{r}} \cdots \otimes_{\Lambda/\mathfrak{r}} (\oplus_{i,j \in I} D_i \otimes_k {}_iM_j \otimes_k D_j) \\ &\simeq \oplus_{i_0, i_1, \dots, i_n \in I} (D_{i_0} \otimes_k {}_{i_0}M_{i_1} \otimes_k D_{i_1} \otimes_k {}_{i_1}M_{i_2} \otimes_k \cdots \otimes_k D_{i_{n-1}} \otimes_k {}_{i_{n-1}}M_{i_n} \otimes_k D_{i_n}) \end{aligned}$$

If k is an algebraically closed field then $D_i \simeq k$ for all $i \in I$, hence

$$\tilde{M} = \oplus_{i,j \in I} D_j \otimes_k {}_jM_i \otimes_k D_i \simeq \oplus_{i,j \in I} k \otimes_k {}_jM_i \otimes_k k \simeq \oplus_{i,j \in I} ({}_jM_i) = M = \mathfrak{r}/\mathfrak{r}^2$$

Hence in the algebraically closed case $\tilde{\mathcal{S}}_\Lambda = \mathcal{S}_\Lambda$ and $\tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) = T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)$.

The next proposition is similar to Proposition 3.2.

Proposition 4.1. *Assume that Λ is a finite dimensional basic split k -algebra.*

- (a) *Let $\{r_1, r_2, \dots, r_m\}$ be elements in \mathfrak{r} such that their images $\{\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m\}$ in $\mathfrak{r}/\mathfrak{r}^2$ generate $\mathfrak{r}/\mathfrak{r}^2$ as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. Then $\{D_1, D_2, \dots, D_n, r_1, r_2, \dots, r_m\}$ generate Λ as a k -algebra, where $\Lambda/\mathfrak{r} \simeq \oplus_{i=1}^n D_i$.*
- (b) *There is a surjective k -algebra homomorphism $\tilde{f}: \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ such that $\oplus_{j \geq \text{rl}(\Lambda)} \tilde{M}^{(j)} \subset \ker \tilde{f} \subset \oplus_{j \geq 1} \tilde{M}^{(j)}$.*

Proof. (a) Similar to Proposition 3.2.

- (b) Fix a map $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon \simeq \text{id}_D$ for the natural projection $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$, and let g be as described above this proposition. From the earlier discussion g lifts to a composition ph where $h: \tilde{M} \rightarrow \mathfrak{r}$ is a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule morphism and p is the natural projection $\mathfrak{r} \twoheadrightarrow M$ viewed as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule morphism via the fixed lifting ϵ .

Construct the morphism $f: \Lambda/\mathfrak{r} \oplus \tilde{M} \rightarrow \Lambda$ by letting $f|_{\Lambda/\mathfrak{r}} = \epsilon$ and $f|_{\tilde{M}} = ih$ where i is the inclusion $\mathfrak{r} \hookrightarrow \Lambda$ viewed as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule morphism via the lifting ϵ . Then, since ϵ is a k -algebra homomorphism and ih is a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule morphism via ϵ , we can use Lemma 3.1 to find a k -algebra homomorphism $\tilde{f}: \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \simeq T(\Lambda/\mathfrak{r}, \tilde{M}) \rightarrow \Lambda$. By part (a) this morphism is surjective.

Since $\tilde{f}|_{\Lambda/\mathfrak{r}} = f|_{\Lambda/\mathfrak{r}} = \epsilon$ is a monomorphism the image intersects trivially with \mathfrak{r} . Using $\tilde{f}(\tilde{M}) \subset \mathfrak{r}$, we then get $\ker \tilde{f} \subset \bigoplus_{j \geq 1} \tilde{M}^{(j)}$. On the other hand, since $\tilde{f}(\tilde{M}^{(j)}) \subset \mathfrak{r}^j$ and $\mathfrak{r}^{\text{rl}(\Lambda)} = (0)$, we get that $\ker \tilde{f} \supset \bigoplus_{j \geq \text{rl}(\Lambda)} \tilde{M}^{(j)}$. \square

Even though this result is similar to Proposition 3.2, and then also to [ARS, Theorem III.1.9], there is an important difference. In [ARS, Theorem III.1.9 (b)] one assumes Λ to be a finite dimensional basic k -algebra where k is algebraically closed, and shows there is a surjective k -algebra homomorphism $\tilde{f}': T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ with $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^j \subset \ker \tilde{f}' \subset \bigoplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^j$. This is a special case of Proposition 3.2(b), where one assumes that Λ is a finite dimensional basic \mathfrak{r} -split k -algebra, and shows there is a surjective k -algebra homomorphism $\tilde{f}'': T(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ with $\bigoplus_{j \geq \text{rl}(\Lambda)} (\mathfrak{r}/\mathfrak{r}^2)^j \subset \ker \tilde{f}'' \subset \bigoplus_{j \geq 2} (\mathfrak{r}/\mathfrak{r}^2)^j$. Hence Proposition 4.1 differs from these two results since the kernel of \tilde{f}' and \tilde{f}'' sits inside $\bigoplus_{j \geq 2} \mathfrak{r}/\mathfrak{r}^2$, while $\ker(\tilde{f}) \subset \bigoplus_{j \geq 1} \tilde{M}^j$ (note which sets the direct sum is taken over). This difference is not surprising, since \tilde{M} usually is a larger module than $\mathfrak{r}/\mathfrak{r}^2$. If k is an algebraically closed field, then $\tilde{M} \simeq \mathfrak{r}/\mathfrak{r}^2$, so $\tilde{f}|_{\tilde{M}} = \tilde{f}'|_{\mathfrak{r}/\mathfrak{r}^2}: \mathfrak{r}/\mathfrak{r}^2 \rightarrow f(\mathfrak{r}/\mathfrak{r}^2) = f(\tilde{M})$ is a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule monomorphism intersecting trivially both with the image of $\tilde{f}|_D$ and \mathfrak{r}^2 , so we get $\ker(\tilde{f}) \subset \bigoplus_{j \geq 2} \tilde{M}^j$. This shows that [ARS, Theorem III.1.9 (b)] is a special case of Proposition 4.1(b). On the other hand, Proposition 3.2(b) is not a special case of Proposition 4.1(b).

The following proposition is similar to Proposition 3.3.

Proposition 4.2. *Let Λ be a finite dimensional basic split k -algebra, let $\tilde{\mathcal{S}}_\Lambda$ be the enlarged species of Λ , and let \tilde{J} be the ideal $\bigoplus_{j \geq 1} \tilde{M}^j$ of $T(\tilde{\mathcal{S}}_\Lambda)$. Then $\Lambda \simeq T(\tilde{\mathcal{S}}_\Lambda)/\langle \rho \rangle$ for a set of relations ρ such that $\tilde{J}^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset \tilde{J}$.*

Proof. The proof is similar to the proof of Proposition 3.3. First recall that $T(\tilde{\mathcal{S}}_\Lambda) = T(\Lambda/\mathfrak{r}, \tilde{M}) = \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)$. From Proposition 4.1 we have an epimorphism $\tilde{f}: T(\tilde{\mathcal{S}}_\Lambda) = \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ with $\bigoplus_{j \geq \text{rl}(\Lambda)} \tilde{M}^j \subset \ker \tilde{f} \subset \bigoplus_{j \geq 1} \tilde{M}^j$. It is possible to find a set of relations $\rho = \{\sigma_t\}_{t \in T}$ in $\ker \tilde{f}$ which generates $\ker \tilde{f}$ as an ideal in $T(\tilde{\mathcal{S}}_\Lambda)$. Then $\tilde{J}^{\text{rl}(\Lambda)} = \bigoplus_{j \geq \text{rl}(\Lambda)} \tilde{M}^j \subset \langle \rho \rangle \subset \bigoplus_{j \geq 1} \tilde{M}^j = \tilde{J}$. Since $T(\tilde{\mathcal{S}}_\Lambda)/\langle \rho \rangle \simeq \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)/\ker \tilde{f} \simeq \Lambda$, we are done. \square

Using Corollary 2.4 and Proposition 4.2 we get the following corollary.

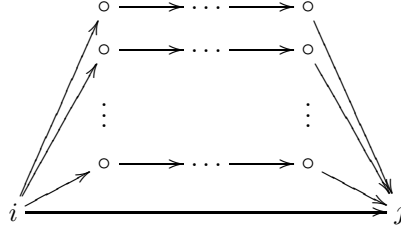
Corollary 4.3. *Let Λ be a finite dimensional basic split k -algebra. Then the category $\text{mod } \Lambda$ is equivalent to the category $\text{rep}(\tilde{\mathcal{S}}_\Lambda, \rho)$ where ρ is a set of relations such that $\tilde{J}^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset \tilde{J}$.*

In the case when Λ is \mathfrak{r} -split, then $\Lambda \simeq T(\mathcal{S}_\Lambda)/\langle \rho \rangle$ where $\langle \rho \rangle$ is an *admissible ideal*, i.e. there exists a natural number $n \geq 2$ such that $J^n \subset \langle \rho \rangle \subset J^2$. On the other hand, if Λ is only split the ideal $\langle \rho \rangle$ is no longer admissible in general. This difference will play an important role when investigating hereditary algebras in the next sections.

5. HEREDITARY BASIC FINITE DIMENSIONAL SPLIT ALGEBRAS

In this section we will continue looking at basic finite dimensional split algebras that are not assumed to be \mathfrak{r} -split, but with the extra assumption that they are hereditary. In contrast to the \mathfrak{r} -split algebras, the species of these hereditary algebras might have non-zero relations. However, the relations are corresponding to subquivers for which the underlying graph contains cycles. We start by introducing these subquivers.

A quiver consisting of an arrow $i \rightarrow j$ together with a finite number (possibly zero) of paths between i and j of length greater than one will be called a *canonical quiver*.



We call the arrow $i \rightarrow j$ in a canonical quiver the *specifying arrow of the canonical quiver*. If Q is a finite quiver without oriented cycles or double arrows, and $i \rightarrow j$ is the specifying arrow of a canonical quiver $Q' \subset Q$, then there exists a largest canonical quiver $Q'' \subset Q$ having $i \rightarrow j$ as its specifying arrow. We say that Q'' is the canonical quiver corresponding to $i \rightarrow j$ in Q . Note that a canonical quiver can be equal to its specifying arrow. If $Q_{\tilde{S}}$ is the quiver of an enlarged species \tilde{S}_{Λ} of a finite dimensional hereditary basic split algebra Λ , then $Q_{\tilde{S}}$ is finite without oriented cycles and without double arrows, so in this setting we always have a largest canonical quiver corresponding to any given arrow.

Let $\mathcal{S} = (D_{i,j}M_i)_{i,j \in I}$ be a species for which the underlying quiver $Q_{\mathcal{S}}$ is finite and without oriented cycles, and let ${}_j\sigma_i = g_1 + \dots + g_n \in {}_jM_i = D_iT(\mathcal{S})D_j$ be a relation. If ${}_j\sigma_i \notin J^2$ where $J = \bigoplus_{i \geq 1} M^{(i)}$, there must be an arrow $i \rightarrow j \in Q_{\mathcal{S}}$ and at least one $g_l \in {}_jM_i$. If there is a set $\{g_l\}$ in ${}_jM_i$, say $\{g_1, \dots, g_m\}$ where $g_i \in {}_jM_i$ and $m \leq n$, letting $\sigma' = g'_1 + g_{m+1} + \dots + g_n$ where $g'_1 = g_1 + \dots + g_m$ we see that $T(\mathcal{S})/\langle \sigma \rangle \simeq T(\mathcal{S})/\langle \sigma' \rangle$ since $\sigma = \sigma'$. Hence we can always reduce to the case with only one $g_l \in {}_jM_i$.

If $\mathcal{S} = (D_{i,j}M_i)_{i,j \in I}$ is a species, where the underlying quiver $Q_{\mathcal{S}}$ is finite and without oriented cycles, and ${}_j\sigma_i = g_1 + \dots + g_n$ is a relation where $g_l \in {}_jM_i$ while $g_k \notin {}_jM_i$ for all $k \neq l$, then we call the relation ${}_j\sigma_i$ a *canonical relation*. By renumbering we will always assume $l = 1$ for a canonical relation. If $n > 1$ we call σ a *strong canonical relation*. We call ρ a *canonical set of relations* and $\langle \rho \rangle$ a *canonical ideal* of $T(\mathcal{S})$ if $\rho = \{\sigma_t\}_{t \in T}$ such that

- (i) all σ_t are canonical relations
- (ii) if $\{\sigma_t\}_{t \in T'}$ is the set of relations in ρ which start in i and end in j , and $\{g_1^t\}_{t \in T'}$ is the corresponding set of summands which are elements of ${}_jM_i$, then $\langle g_1^t \rangle \cap \langle g_1^{t'} \rangle = (0)$ for all $t \neq t'$

If all the canonical relations are strong, we call ρ a *strong canonical set of relations* and $\langle \rho \rangle$ a *strong canonical ideal*.

Lemma 5.1. *Let \mathcal{S} be a species for which the underlying quiver $Q_{\mathcal{S}}$ is finite and without oriented cycles, and let ρ be a canonical set of relations. Then $T(\mathcal{S})/\langle \rho \rangle$ is hereditary.*

Proof. Let $\Lambda = T(\mathcal{S})/\langle \rho \rangle$ be as described in the lemma, then every indecomposable projective Λ -module is of the form $P_i \simeq \Lambda D_i$ where $\Lambda/\mathfrak{r} \simeq \bigoplus_{i \in I} D_i$. We want to show

that the radical $\mathfrak{r}D_i$ of every projective module ΛD_i is again projective. Assume $\{{}_j\sigma_i^t\}_{t \in {}_jT_i}$ is the complete set of (canonical) relations starting in i and ending in j , and let ${}_jN_i = {}_jM_i / \langle \{g_1^t\}_{t \in {}_jT_i} \rangle$ where ${}_j\sigma_i^t = g_1^t + g_2^t + \dots + g_{n_i}^t$ and $g_1^t \in {}_jM_i$. Let I' be the set of vertices $j \in I$ for which ${}_jM_i \neq (0)$, and let $I' = I_c \cup I_n$ be the disjoint union where $j \in I_c$ if there exists a (canonical) relation ${}_j\sigma_i$, and $j \in I_n$ if no such relation exists. Then

$$\begin{aligned} \mathfrak{r}D_i &= (\text{rad } T(\mathcal{S})D_i) / (\langle \rho \rangle D_i) \\ &\simeq (\oplus_{j \in I_c} (T(\mathcal{S}){}_jN_i) / (\langle \rho \rangle {}_jN_i)) \oplus (\oplus_{j \in I_n} (T(\mathcal{S}){}_jM_i) / (\langle \rho \rangle {}_jM_i)) \\ &\simeq (\oplus_{j \in I_c} (\dim_{D_j} {}_jN_i) (T(\mathcal{S})D_j) / (\langle \rho \rangle D_j)) \oplus (\oplus_{j \in I_n} (\dim_{D_j} {}_jM_i) (T(\mathcal{S})D_j) / (\langle \rho \rangle D_j)) \\ &= (\oplus_{j \in I_c} (\dim_{D_j} {}_jN_i) \Lambda D_j) \oplus (\oplus_{j \in I_n} (\dim_{D_j} {}_jM_i) \Lambda D_j) \\ &\simeq (\oplus_{j \in I_c} (\dim_{D_j} {}_jN_i) P_j) \oplus (\oplus_{j \in I_n} (\dim_{D_j} {}_jM_i) P_j) \end{aligned}$$

This shows that $\mathfrak{r}D_i$ is projective, hence $\mathfrak{r} = \oplus_{i \in I} \mathfrak{r}D_i$ is projective, which implies that Λ is hereditary [La2, Theorem 2.35]. \square

Let Λ be a finite dimensional hereditary basic split k -algebra, where $\Lambda \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho \rangle$ and $Q_{\tilde{\mathcal{S}}_\Lambda}$ is the underlying quiver of $\tilde{\mathcal{S}}_\Lambda$. From the discussion after Proposition 3.3 we see that we can choose the set of relations $\rho = \{\sigma_t\}_{t \in T}$ finite, therefore we can find a finite minimal set of relations which generate $\langle \rho \rangle$. The next proposition reveals that even in the non \mathfrak{r} -split case we find interesting information on the ideal $\langle \rho \rangle$.

Proposition 5.2. *Let Λ be a finite dimensional hereditary basic split k -algebra, where $\Lambda \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho \rangle$ and ρ is a finite set of relations. If $\rho = \{\sigma_1, \dots, \sigma_m\}$ is a minimal set, then ρ is a canonical set of relations.*

Proof. Let $\rho = \rho' \cup \rho''$ where ρ' is the subset of ρ consisting of canonical relations, and $\rho'' = \rho \setminus \rho'$. If there exists a pair $\sigma_t, \sigma_{t'} \in \rho'$ of relations starting in i and ending in j such that $\langle g_1^t \rangle \cap \langle g_1^{t'} \rangle \neq (0)$, then $g_1^{t'} = d_j g_1^t d_i$ for $d_i \in D_i$ and $d_j \in D_j$, so by substitution we can find $\sigma_{t'}' \in \tilde{J}^2$ such that $T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho \rangle \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \sigma_t', \rho \setminus \sigma_{t'} \rangle$. By repeating this process we find a set $\varrho = \varrho' \cup \varrho''$ of relations where every element of ϱ' is canonical, every element of ϱ'' sits inside \tilde{J}^2 , and for every pair $\sigma_t, \sigma_{t'} \in \varrho'$ we have $\langle g_1^t \rangle \cap \langle g_1^{t'} \rangle = (0)$, i.e. ϱ' is a canonical set of relations. Let $\Lambda' \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \varrho' \rangle$, then Λ' is hereditary from Lemma 5.1.

Now

$$\begin{aligned} \Lambda &\simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \varrho \rangle \simeq \Lambda' / (\langle \varrho \rangle / \langle \varrho' \rangle) \\ &\simeq \Lambda' / (\langle \varrho'' \rangle / (\langle \varrho'' \rangle \cap \langle \varrho' \rangle)) \end{aligned}$$

Since $\langle \varrho'' \rangle \subset \tilde{J}^2$ we have $\langle \varrho'' \rangle / (\langle \varrho'' \rangle \cap \langle \varrho' \rangle) \subset \tilde{J}^2 / \langle \varrho' \rangle \simeq \text{rad}^2 \Lambda'$ as left Λ' -modules. Lemma 3.5 implies $\langle \varrho'' \rangle = 0$ since Λ was assumed to be hereditary, hence $\varrho = \varrho'$. Since $\rho'' \subset \varrho'' = \emptyset$, we have $\rho = \rho'$. Moreover the existence of a pair $\sigma_t, \sigma_{t'} \in \rho'$ of relations starting in i and ending in j such that $\langle g_1^t \rangle \cap \langle g_1^{t'} \rangle \neq (0)$ implies $\varrho'' \neq \emptyset$ since ρ was assumed to be minimal, a contradiction. Hence ρ is a canonical set of relations. \square

Proposition 5.3. *Let Λ be a finite dimensional hereditary basic split k -algebra. Then $\Lambda \simeq T(\mathcal{S}_m) / \langle \rho \rangle$ where \mathcal{S}_m is a species unique up to isomorphism for which the underlying quiver $Q_{\mathcal{S}_m}$ is finite and without oriented cycles, and $\langle \rho \rangle$ is a strong canonical ideal.*

Proof. From Proposition 4.2 we know that $\Lambda \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho' \rangle$ where we can choose the set ρ' to be finite, and from Lemma 5.2 we know that ρ' is a canonical set of relations.

Let $\sigma = g_1 + \cdots + g_n \in \rho$ be a relation where $g_1 \in {}_j\tilde{M}_i$. If the largest canonical quiver containing $i \rightarrow j$ is the trivial canonical quiver $i \rightarrow j$ itself, then ${}_j\tilde{\mathcal{M}}_i = {}_j\tilde{M}_i$ and ${}_j f_i: {}_j\tilde{\mathcal{M}}_i \otimes_{D_i} V_i \rightarrow V_j$ is just the morphism ${}_j\phi_i: {}_j\tilde{M}_i \otimes_{D_i} V_i \rightarrow V_j$. This implies that $\sigma = g_1 \in {}_j\tilde{M}_i$, so $\langle \sigma \rangle$ generated as a $D_j - D_i$ -bimodule is a submodule of ${}_j\tilde{M}_i$. If we let ${}_j\tilde{N}_i = {}_j\tilde{M}_i / \langle \sigma \rangle$, we can define a species \mathcal{S}_σ by substituting the bimodule ${}_j\tilde{M}_i$ in $\tilde{\mathcal{S}}_\Lambda$ with ${}_j\tilde{N}_i$. From the construction of the tensor algebra we see that $T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho' \rangle \simeq T(\mathcal{S}_\sigma) / \langle \rho' \setminus \sigma \rangle$.

If we use this process repeatedly, we can remove all relations in ρ' that are not strong canonical relations, and end up with a unique (up to isomorphism) species \mathcal{S}_m together with a set ρ consisting of the strong canonical relations in ρ' such that $\Lambda \simeq T(\mathcal{S}_m) / \langle \rho \rangle$. \square

Theorem 5.4. *Let Λ be a finite dimensional basic split k -algebra. Then the following are equivalent*

- (i) Λ is hereditary.
- (ii) $\Lambda \simeq T(\mathcal{S}) / \langle \rho \rangle$ where \mathcal{S} is a species for which the underlying quiver $Q_{\mathcal{S}}$ is finite and without oriented cycles and $\langle \rho \rangle$ is a canonical ideal.
- (iii) $\Lambda \simeq T(\mathcal{S}_m) / \langle \rho' \rangle$ where \mathcal{S}_m is a species unique up to isomorphism for which the underlying quiver $Q_{\mathcal{S}_m}$ is a subquiver of $Q_{\mathcal{S}}$ and $\langle \rho' \rangle$ is a strong canonical ideal.

Proof. This follows from Lemma 5.1, Proposition 5.2, and Proposition 5.3. Note that $Q_{\mathcal{S}_m}$ might be a strict subquiver if a bimodule ${}_j\tilde{M}_i$ is replaced by a bimodule ${}_j\tilde{N}_i = (0)$. \square

The next theorem sums up the results in this section similarly to what Theorem 3.12 did for Section 3.

Theorem 5.5. *Let Λ be a finite dimensional basic split k -algebra, let $\tilde{\mathcal{S}}_\Lambda = (D_i, {}_j\tilde{M}_i)_{i,j \in I}$ be the enlarged species of Λ , let $Q_{\tilde{\mathcal{S}}_\Lambda}$ be the underlying quiver of $\tilde{\mathcal{S}}_\Lambda$, and let \tilde{J} be the ideal $\oplus_{i \geq 1} \tilde{M}^{(i)}$ in $\tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2)$ where $\tilde{M} = \oplus_{i,j \in I} {}_j\tilde{M}_i$. Then the following hold:*

- (a) *Let $\{r_1, r_2, \dots, r_m\}$ be elements in \mathfrak{r} such that their images $\{\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m\}$ in $\mathfrak{r}/\mathfrak{r}^2$ generate $\mathfrak{r}/\mathfrak{r}^2$ as a $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule. Then $\{D_1, D_2, \dots, D_n, r_1, r_2, \dots, r_m\}$ generate Λ as a k -algebra, where $\Lambda/\mathfrak{r} \simeq \oplus_{i=1}^n D_i$.*
- (b) *There is an onto k -algebra homomorphism $\tilde{f}: \tilde{T}(\Lambda/\mathfrak{r}, \mathfrak{r}/\mathfrak{r}^2) \rightarrow \Lambda$ such that $\oplus_{j \geq \text{rl}(\Lambda)} \tilde{M}^{(j)} \subset \ker \tilde{f} \subset \oplus_{j \geq 1} \tilde{M}^{(j)}$.*
- (c) *$\Lambda \simeq T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho \rangle$ where ρ is a set of relations on $\tilde{\mathcal{S}}$ with $\tilde{J}^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset \tilde{J}$.*
- (d) *The category $\text{mod } \Lambda$ is equivalent to the category $\text{rep}(\tilde{\mathcal{S}}_\Lambda, \rho)$, where ρ is a set of relations on $\tilde{\mathcal{S}}_\Lambda$ with $\tilde{J}^{\text{rl}(\Lambda)} \subset \langle \rho \rangle \subset \tilde{J}$.*
- (e) *Λ is hereditary if and only if $Q_{\tilde{\mathcal{S}}_\Lambda}$ is finite without oriented cycles and Λ is isomorphic to $T(\tilde{\mathcal{S}}_\Lambda) / \langle \rho \rangle$ for a canonical ideal ρ .*
- (f) *Λ is hereditary if and only if Λ is isomorphic to $T(\mathcal{S}_m) / \langle \rho \rangle$ for a species \mathcal{S}_m and a strong canonical ideal ρ where the underlying quiver $Q_{\mathcal{S}_m}$ of \mathcal{S}_m is finite and without oriented cycles.*

Let Λ be a finite dimensional hereditary basic split k -algebra, and let $\Lambda \simeq T(\mathcal{S}_m) / \langle \rho \rangle$ where $\mathcal{S}_m = (D_i, {}_jN_i)_{i,j \in I}$ and ρ are as described in Proposition 5.3, hence ρ consists of strong canonical relations. We want to investigate the species \mathcal{S}_m . As before, let $\Lambda/\mathfrak{r} \simeq \oplus D_i = D$ and $\mathfrak{r}/\mathfrak{r}^2 \simeq \oplus_{i,j \in I} ({}_jM_i) = M$. Let $i \rightarrow j$ be a trivial canonical quiver. We obviously have an isomorphism $D_j \Lambda D_i \simeq D_j (T(\mathcal{S}_m) / \langle \rho \rangle) D_i$. What is interesting is that $D_j \langle \rho \rangle D_i = (0)$ since $i \rightarrow j$

was a trivial canonical quiver and ρ only consists of strong canonical relations. Therefore $D_j \Lambda D_i \simeq D_j(T(\mathcal{S}_m)/\langle \rho \rangle) D_i \simeq D_j T(\mathcal{S}_m) D_i$.

If the underlying graph of \mathcal{S}_m is a tree, i.e. it does not contain any cycles, then all canonical quivers in \mathcal{S}_m must be trivial. This yields the following corollary.

Corollary 5.6. *If the underlying graph of \mathcal{S}_m is a tree, then $\Lambda \simeq T(\mathcal{S}_m) \simeq T(\mathcal{S}_\Lambda)$.*

This corollary can also be deduced from the following observation [DR2]: Let Λ be a finite dimensional hereditary basic split k -algebra, and let $Q_{\mathcal{S}_\Lambda}$ be the underlying quiver of the species associated to Λ . Assume $Q_{\mathcal{S}_\Lambda}$ is a tree. Then for any pair $i, j \in I$, either $D_j \mathfrak{r}^2 D_i$ or $D_j \mathfrak{r}/\mathfrak{r}^2 D_i$ must be zero. Then obviously

$$0 \rightarrow D_j \mathfrak{r}^2 D_i \rightarrow D_j \mathfrak{r} D_i \rightarrow D_j \mathfrak{r}/\mathfrak{r}^2 D_i \rightarrow 0$$

splits as a sequence of $D_j - D_i$ -modules via any k -algebra homomorphism $\epsilon: \Lambda/\mathfrak{r} \rightarrow \Lambda$ such that $\pi\epsilon \simeq \text{id}_{\Lambda/\mathfrak{r}}$ for the natural projection $\pi: \Lambda \rightarrow \Lambda/\mathfrak{r}$. This implies that

$$0 \rightarrow \mathfrak{r}^2 \rightarrow \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{r}^2 \rightarrow 0$$

splits as a sequence of $\Lambda/\mathfrak{r} - \Lambda/\mathfrak{r}$ -bimodule via ϵ . Hence Λ is \mathfrak{r} -split, so $\Lambda \simeq T(\mathcal{S}_\Lambda)$.

Example 5.7. We will end this article with an example of a hereditary species containing canonical relations. This example is motivated by [DR2]. Let $k = \mathbb{F}_2(t^2) = \{\frac{f}{g} \mid f, g \in \mathbb{F}_2[t^2], g \neq 0\}$ be the field of rational functions with indeterminate t^2 over the ground field \mathbb{F}_2 , where \mathbb{F}_2 is the Galois field consisting of two elements. The field k is not perfect. Let $K = \mathbb{F}_2(t)$. We define a morphism $\delta: K \rightarrow K$ by using the usual derivation with respect to t . Now $\delta(f) = 0$ for $f \in k \subset K$ due to the fact that $\text{char } \mathbb{F}_2 = 2$. Let M be the set $M = \{(f, g) \mid f, g \in K\}$ where we define a $K - K$ -bimodule structure on M by letting $a(f, g) = (af, ag)$ and $(f, g)b = (fb, gb + f\delta(b))$ for $(f, g) \in M$ and $a, b \in K$. The species \mathcal{S} is given by the following diagram

$$\begin{array}{ccc} & K & \\ K \nearrow & & \searrow K \\ & M & \\ K \xrightarrow{\quad} & & K \end{array}$$

where the underlying quiver $Q_{\mathcal{S}}$ of \mathcal{S} is

$$\begin{array}{ccc} & 1 & \\ 2 \nearrow & & \searrow \\ & 0 & \\ 2 \xrightarrow{\quad} & & 0 \end{array}$$

Let σ be the relation $((1, 0), 0) - (0, 1 \otimes_K 1) \in {}_0\mathcal{M}_2 = M \oplus (K \otimes_K K)$, and let ρ be the set of relations consisting only of σ . Then (\mathcal{S}, ρ) is a species with relations, and ρ is a set of strong canonical relations. The tensor ring $T(\mathcal{S})/\langle \rho \rangle$ is isomorphic to the matrix ring

$$\Lambda = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ M & K & K \end{pmatrix}$$

where multiplication is normal matrix multiplication except for the following

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (ab, 0) & 0 & 0 \end{pmatrix}$$

From [DR2, Corollary 2] we know that this is a hereditary finite dimensional k -

algebra. We have $\Lambda/\tau \simeq \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$, and since

$$\epsilon: \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix} \hookrightarrow \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ M & K & K \end{pmatrix}$$

is a k -algebra homomorphism such that $\pi\epsilon \simeq \text{id}_{\Lambda/\tau}$ for the natural projection $\pi: \Lambda \rightarrow \Lambda/\tau$, we see that Λ is split. Look at the sequence

$$0 \rightarrow D_0\tau^2 D_2 \rightarrow D_0\tau D_2 \rightarrow D_0\tau/\tau^2 D_2$$

of $\Lambda/\tau - \Lambda/\tau$ -bimodules via ϵ . This is the sequence

$$0 \rightarrow K \rightarrow M \rightarrow K \rightarrow 0$$

where the first morphism is given by $a \mapsto (a, 0)$ for $a \in K$, and the last morphism is given by $(a, b) \mapsto b$ for $(a, b) \in M$. Since $M \not\cong K^2$ this sequence does not split, hence Λ is not a τ -split algebra. Hence Λ is an example of an algebra that satisfies the assumptions in Theorem 5.4, but not the assumptions in Theorem 3.9. \triangle

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